INTERNATIONAL BACCALAUREATE
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MARKSCHEME

May 2000

MATHEMATICS

Higher Level

Paper 2

30 pages

1. (a) Using the formula for the area of a triangle gives

$$A = \frac{1}{2}x\,3x\sin\theta$$ ***(M1)***

$$\sin\theta = \frac{4.42}{3x^2}$$ ***(A1)***

[2 marks]

(b) Using the cosine rule gives

$$\cos\theta = \frac{x^2 + (3x)^2 - (x+3)^2}{2 \times x \times 3x}$$ ***(M1)***

$$= \frac{3x^2 - 2x - 3}{2x^2}$$ ***(A1)***

[2 marks]

(c) (i) Substituting the answers from (a) and (b) into the identity $\cos^2\theta = 1 - \sin^2\theta$ gives ***(M1)***

$$\left(\frac{3x^2 - 2x - 3}{2x^2}\right)^2 = 1 - \left(\frac{4.42}{3x^2}\right)^2$$ ***(AG)***

(ii) (a) $x = 1.24, 2.94$ ***(G1)(G1)***

(b) $\theta = \arccos\left(\frac{3x^2 - 2x - 3}{2x^2}\right)$ ***(M1)***

$\theta =$ 1.86 radians or $\theta = 0.171$ (accept 0.172) radians (3 s.f.) ***(G1)(G1)***

> **Notes:** Some calculators may not produce answers that are as accurate as required, especially if they use 'zoom and trace' to find the answers. Allow ±0.02 difference in the value of x, with appropriate **ft** for θ.
>
> Award ***(M1)(G1)(G0)*** for correct answers given in degrees (106° or 9.84°).
> Award ***(M1)(G1)(G0)*** if the answers are not given to 3 s.f.
> Award ***(M0)(G2)*** for correct answers without working.

[6 marks]

[Total: 10 marks]

2. (a) Let $g(x) = ax^3 + bx^2 + cx + d$

$g(0) = -4 \Rightarrow d = -4$ ***(A1)***

$g'(x) = 3ax^2 + 2bx + c$ ***(M1)***

$g'(0) = 0 \Rightarrow c = 0$ ***(A1)***

$g(-2) = 0 \Rightarrow -8a + 4b = 4$

$g'(-2) = 0 \Rightarrow 12a - 4b = 0$ ***(M1)***

$4a = 4$

$a = 1$ ***(A1)***

$b = 3$ ***(A1)***

Therefore, $g(x) = x^3 + 3x^2 - 4$ ***(AG)***

[6 marks]

(b) Under reflection in the *y*-axis, the graph of $y = -x^3 + 3x^2$ is mapped onto the graph of

$y = -(-x)^3 + 3(-x)^2$ ***(M1)***

i.e. $y = x^3 + 3x^2$. ***(A1)***

Under translation $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$, the graph of $y = x^3 + 3x^2$ is mapped onto the graph of

$y = h(x) = (x+1)^3 + 3(x+1)^2 - 1$ ***(M1)***

$= x^3 + 3x^2 + 3x + 1 + 3x^2 + 6x + 3 - 1$ ***(A1)***

$h(x) = x^3 + 6x^2 + 9x + 3$ ***(A1)***

[5 marks]

(c) The graph of $y = -x^3 + 3x^2$ is mapped onto the graph of $y = x^3 + 3x^2 - 4$, with point *A* mapped onto point A', using the following combination of transformations:

Reflection in the *x*-axis ***(A1)***

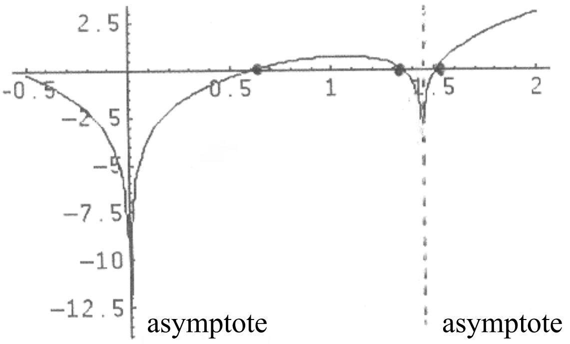
followed by the translation $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$. ***(A2)***

(or vice versa.)

[3 marks]

[Total: 14 marks]

3. (a) (i)



(G2)

Award ***(G1)*** for correct shape, including three zeros, and ***(G1)*** for both asymptotes

(ii) $f(x) = 0$ for $x = 0.599, 1.35, 1.51$ ***(G1)(G1)(G1)***

[5 marks]

(b) $f(x)$ is undefined for

$(x^5 - 3x^2) = 0$ ***(M1)***

$x^2(x^3 - 3) = 0$

Therefore, $x = 0$ or $x = 3^{1/3}$ ***(A2)***

[3 marks]

(c) $f'(x) = \dfrac{5x^4 - 6x}{x^5 - 3x^2} \left(\text{or } \dfrac{5x^3 - 6}{x^4 - 3x} \right)$ ***(M1)(A1)***

$f'(x)$ is undefined at $x = 0$ and $x = 3^{1/3}$ ***(A1)***

[3 marks]

(d) For the x-coordinate of the local maximum of $f(x)$, where $0 < x < 1.5$ put $f'(x) = 0$ ***(R1)***

$5x^3 - 6 = 0$ ***(M1)***

$x = \left(\dfrac{6}{5} \right)^{\frac{1}{3}}$ ***(A1)***

[3 marks]

(e) The required area is

$$A = \int_{0.599}^{1.35} f(x)\,\mathrm{d}x$$ ***(A2)***

Note: Award ***(A1)*** for each correct limit

[2 marks]

[Total: 16 marks]

4.

Note:	In all 3 parts, award ***(A2)*** for correct answers with no working. Award ***(M2)(A2)*** for correct answers with written evidence of the correct use of a GDC (see **GDC examples**).

(a) (i) Let X be the random variable "the weight of a bag of salt". Then $X \sim \mathrm{N}(110, \sigma^2)$, where σ is the new standard deviation.

Given $\mathrm{P}(X < 108) = 0.07$, let $Z = \dfrac{X - 110}{\sigma}$

Then $\mathrm{P}\left(Z < \dfrac{108 - 110}{\sigma}\right) = 0.07$ ***(M1)***

Therefore, $\dfrac{-2}{\sigma} = -1.476$ ***(M1)(A1)***

Therefore, $\sigma = 1.355$ ***(A1)***

GDC Example:	Graphing of normal c.d.f. with σ as the variable, and finding the intersection with $p = 0.07$.

[4 marks]

(ii) Let the new mean be μ, then $X \sim \mathrm{N}(\mu, 1.355^2)$.

Then $\mathrm{P}\left(Z < \dfrac{108 - \mu}{1.355}\right) = 0.04$ ***(M1)***

Therefore, $\dfrac{108 - \mu}{1.355} = -1.751$ ***(M1)(A1)***

Therefore, $\mu = 110.37$ ***(A1)***

GDC Example:	Graphing of normal c.d.f. with μ as the variable and finding intersection with $p = 0.04$.

[4 marks]

continued...

Question 4 continued

(b) If the mean is 110.37 g then $X \sim \mathrm{N}(110.37, 1.355^2)$, $\mathrm{P}(A < X < B) = 0.8$

Then $\mathrm{P}(X < A) = 0.1$, and $\mathrm{P}(X < B) = 0.9$.

Therefore, $\dfrac{A - 110.37}{1.355} = -1.282$ ***(M1)***

$A = 108.63$ ***(A1)***

Therefore, $\dfrac{B - 110.37}{1.355} = 1.282$ ***(M1)***

$B = 112.11$ ***(A1)***

GDC Example:	Graphing of normal c.d.f. with X as the variable and finding intersection with $p = 0.1$ and 0.9.

[4 marks]

[Total: 12 marks]

5. (i) Using first principles

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\cos(x+h)-\cos(x)}{h}\right)$$ ***(M1)***

$$= \lim_{h\to 0}\left(\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}\right)$$ ***(M1)(A1)***

$$= \cos(x)\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right) - \sin(x)\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)$$ ***(M1)(A1)***

But $\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)=1$ and $\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right)=0$ ***(C1)(C1)***

Therefore, $f'(x) = -\sin x$ ***(A1)***

OR

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\cos(x+h)-\cos(x)}{h}\right)$$ ***(M1)***

$$= \lim_{h\to 0}\left(\frac{-2\sin(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h}\right)$$ (using any method) ***(M1)(A2)***

$$= \lim_{h\to 0}\left(-\sin\left(x+\frac{1}{2}h\right)\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\right)$$ ***(M1)***

But $=\lim_{h\to 0}\left(\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\right)=1$ and $=\lim_{h\to 0}\left(-\sin\left(x+\frac{1}{2}h\right)\right)=-\sin x$ ***(C2)***

Therefore, $f'(x) = -\sin x$ ***(A1)***

[8 marks]

continued...

Question 5 continued

(ii) Let p_n be the statement $\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$ for all positive integer values of n.

If $n = 1$ then $\frac{\mathrm{d}}{\mathrm{d}x}(x^1) = \lim_{k \to 0}\left(\frac{(x+k)-x}{k}\right)$ ***(M1)(A1)***

$= 1$

$= 1x^0$ ***(A1)***

Assume the formula is true for $n = k$, that is, $\frac{\mathrm{d}}{\mathrm{d}x}(x^k) = kx^{k-1}$. ***(M1)***

Then $\frac{\mathrm{d}}{\mathrm{d}x}(x^{k+1}) = \frac{\mathrm{d}}{\mathrm{d}x}(x \times x^k)$ ***(M1)***

$= kx^{k-1} \times x + x^k$ (using the results for $n = k$ and $n = 1$ given above) ***(M1)(A1)***

$= x^k(k+1)$ ***(A1)***

which is the formula when $n = k+1$.

So if the formula is true for $n = k$ then it is true for $n = k+1$. ***(R1)***

p_1 is true, so p_2 is true and p_3 is true, therefore p_n is true for all integer value of n. ***(R1)***

[10 marks]

[Total: 18 marks]

6.

Note:	In this question candidates may use figures to varying degrees of accuracy. Do not penalise, unless premature rounding leads to errors in the final answers, which should be given to 3 s.f.

Let the random variable X represent the test result for hard discs.

$$X \sim \mathrm{N}(68, 3^2)$$

(a) $\mathrm{P}(X < 67) = \mathrm{P}\left(Z < -\frac{1}{3}\right)$ ***(M1)***

$= 1 - 0.6306$

$= 0.369$ (3 d.p.) ***(A1)***

OR

$\mathrm{P}(X < 67) = 0.369$ ***(G2)***

Note:	Award ***(G1)*** for an incorrect answer if there is an indication that the candidate has used the normal cumulative distribution function (cdf).

[2 marks]

(b) $\bar{X} \sim \mathrm{N}\left(68, \frac{3^2}{10}\right)$ ***(M1)***

P(rejection) $= \mathrm{P}(\bar{X} \le 67) = \mathrm{P}\left(Z \le \frac{-1}{\left(3/\sqrt{10}\right)}\right) = \mathrm{P}(Z \le -1.0541)$ ***(M1)***

$= 1 - 0.8541 = 0.1459$ ***(M1)***

$= 0.146$ (3 s.f.) ***(A1)***

OR

Standard error of the mean $= \frac{3}{\sqrt{10}} = 0.9487$ ***(M1)(A1)***

$\mathrm{P}(\bar{X} < 67) = 0.146$ (3 s.f.) ***(G2)***

[4 marks]

continued...

Question 6 continued

(c) (i) let Y be the result for discs from this supplier

$$Y \sim \mathrm{N}(67.5, 2.8^2) \Rightarrow \bar{Y} \sim \mathrm{N}\left(67.5, \frac{2.8^2}{10}\right)$$ ***(M1)***

$$\mathrm{P}(\bar{Y} > 67) = \mathrm{P}\left(Z > \frac{-0.5}{2.8/\sqrt{10}}\right)$$

$$= \mathrm{P}(Z > -0.56469) = 0.7139$$ ***(M1)***

$$= 0.714 \text{ (3 s.f.)}$$ ***(A1)***

[3 marks]

(ii) $\mathrm{P}(\bar{Y} \le 67) = 1 - \mathrm{P}(\bar{Y} > 67) = 1 - 0.7139$ ***(M1)(A1)***

$= 0.2861$ ***(A1)***

Let R be the number of rejections in a week.

$R \sim \mathrm{B}(6, 0.2861)$. ***(M1)***

Hence the expected value of penalty is $1000 \times 6 \times 0.2861$ ***(M1)***

$= \$1716.60$ ***(A1)***

OR

P(rejection on any day) $= 1 - 0.7139 = 0.2861$ ***(A1)***

This is a binomial experiment and the possible outcomes are 0, 1, 2, 3, 4, 5 or 6 rejections. The probabilities of the outcomes are mentined below:

x	0	1	2	3	4	5	6
P(x)	.1324	.3183	.3189	.1704	.0512	.0082	.00055

(A2)

Note: For each error deduct one mark up to a maximum of two marks.

To obtain expected values for each of the above outcomes, the penalties will have to be multiplied by the corresponding probabilities. Thus we have:

Penalty	0	1000	2000	3000	4000	5000	6,000
P(x)	.1324	.3183	.3189	.1704	.0512	.0082	.00055

(A2)

Note: For each error deduct one mark up to a maximum of two marks.

E(Penalty) $= \sum$ (Penalty)(Probability) $= 1716.6$

$= \$1716.60$ ***(A1)***

[6 marks]

continued...

Question 6 continued

(d) Sample mean $\overline{x} = 67.0031$ ***(A1)***

$H_0 : \mu = 68$

$H_1 : \mu < 68$ ***(C1)***

This is lower tail test with $z = -1.645$ ***(C1)***

Decision rule is reject H_0 if $z < -1.645$ ***(C1)***

$$z = \frac{\overline{x} - 68}{\text{standard error}} = \frac{67.0031 - 68}{0.9487} = -1.0508$$ ***(M1)***

Therefore, $z > -1.645$ ***(M1)***

Hence, we do not have enough evidence to reject H_0. Thus the sample meets the company's standard for acceptance. ***(R1)***

OR

Mean of sample $= 67.0031$ ***(A1)***

5 % lower tail critical value $= 68 - 1.645 \times \frac{3}{\sqrt{10}}$ ***(M1)(M1)(M1)***

$= 66.4394$ ***(G2)***

Note: Award ***(M1)*** for 68, ***(M1)*** for 1.645, and ***(M1)*** for the standard error, $\frac{3}{\sqrt{10}}$

This is less than the observed value and hence there is no evidence of change of standards. ***(R1)***

[7 marks]

continued...

Question 6 continued

(e) This is a Chisquare test for the goodness of fit.
The expected frequencies are calculated from the probabilities in the table of normal distributions (or from the calculator) and then multiplied by 1000. We have the following table:

Class	Observed	Probability	Expected	Expected – observed
1	5	0.0013	1.3 }	
2	17	0.0214	21.4 }	0.7
3	146	0.1359	135.9	– 10.1
4	333	0.3413	341.3	– 8.3
5	360	0.3413	341.3	– 18.7
6	113	0.1359	135.9	22.9
7	26	0.0229	22.9	– 3.1

(A3)

Notes: If the first two cells are not combined but all the entries are correct, then award ***(A2)***.
For each error in computing individual entries deduct one mark up to a maximum of two marks.
Do not penalise if the last row is:

7	26	0.0228	22.8	– 3.1

Since the first cell has 5 elements, we combine the first two cells.
This leads to the observed – expected $= 22 - 22.7 = 0.7$.
The number of degrees of freedom is $7 - 1 - 1 = 5$. ***(A1)***
Calculated value of Chisquare $= 6.2771$ ***(A1)***

H_0 : Distribution is normal with mean 68 and standard deviation 3
H_1 : Distribution is not normal with mean 68 and standard deviation 3 ***(C1)***

χ^2 critical value with 5 degrees of freedom is 11.07 ***(A1)***
Since $6.277 < 11.07$, we do not have enough evidence to reject H_0.
Hence, we do not have reason to doubt the normality of the data. ***(R1)***

[8 marks]

7. (i) (a) By definition of • and de Morgan's laws,

$(X \bullet Y)' = (X \cap Y)' \cap (X' \cap Y')'$ ***(M1)***

$= (X' \cup Y') \cap (X \cup Y)$ ***(M1)***

$= (X \cup Y) \cap (X' \cup Y')$ ***(R1)***

[3 marks]

(b) $f(n) = f(n')$, for any n, n' in $\mathbb{N}$, implies $n+1 = n'+1$.
Hence $n = n'$. Hence f is an injection from $\mathbb{N}$ to $\mathbb{N}$. ***(R1)***

There is no point in the domain of f which is mapped to zero. ***(M1)***
Hence f is not a surjection. ***(R1)***

[3 marks]

(c) We show that *S* is a reflexive, symmetric and transitive relation on *X*.
Since *R* is at equivalence relation on *Y*, it is reflexive, symmetric, and transitive.

For all *a* in *X*, reflexivity of *R* implies $h(a) R h(a)$. By the definition of the relation *S* on *X*, $a S a$ for all *a* in *X*. Hence, *S* is reflexive. ***(R1)***

Let $a S b$. Then $h(a) R h(b)$ holds on *Y*. Since *R* is symmetric, $h(b) R h(a)$ which implies $b S a$. Since this holds for all *a*, *b* in *X*.
S is a symmetric relation on *X*. ***(R1)***

Let $a S b$ and $b S c$ for any a, b, c in *X*. Then $h(a) R h(b)$ and $h(b) R h(c)$ hold.
Since *R* is a transitive relation, we get $h(a) R h(c)$. ***(M1)***
By definition of the relation *S* on *X*, $a S c$. Thus *S* is transitive on *X*. ***(R1)***

[4 marks]

(ii) (a) By using the composition of functions we form the Cayley table

$\circ$	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_1	f_4	f_3
f_3	f_3	f_4	f_1	f_2
f_4	f_4	f_3	f_2	f_1

(A3)

Note:	For each error in the above table deduct one mark up to a maximum of three marks.

From the table, we see that $(T, \circ)$ is a closed and $\circ$ is commutative. ***(R1)***

f_1 is the identity. ***(A1)***

$$f_i^{-1} = f_i, i = 1, 2, 3, 4.$$ ***(A1)***

Since the composition of functions is an associative binary operation $(T, \circ)$ is an Abelian group. ***(AG)***

[6 marks]

continued...

Question 7 (ii) continued

(b) The Cayley table for the group $(G, \Diamond)$ is given below:

$\Diamond$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

(A2)

Note: For each error in the entries deduct one mark up to a maximum of two marks.

Define $f : T \mapsto G$ such that $f(f_1) = 1, f(f_2) = 3, f(f_3) = 5$ and $f(f_4) = 7$. ***(M1)***
Since distinct elements are mapped onto distinct images, it is a bijection. ***(R1)***
Since the two Cayley tables match, the bijection is an isomorphism. ***(R1)***
Hence the two groups are isomorphic. ***(AG)***

[5 marks]

(iii) (a) Suppose a is of order n and a^{-1} is of order m.
Therefore $e = e*e = (a^{-1})^m * a^n$ ***(M1)***

If $m > n$, then $e = (a^{-1})^{m-n} * (a^{-1})^n * a^n = (a^{-1})^{m-n} * (a^{-1} * a)^n$. ***(M1)***
Hence $e = (a^{-1})^{m-n}$. This implies a^{-1} is of order $m - n < m$ which is a contradiction. So m is not greater than n. ***(R1)***

If $m < n$, $e = (a^{-1})^m * a^m * a^{n-m} = (a^{-1} * a)^m * a^{n-m}$. ***(M1)***
Hence $e = a^{n-m}$, which implies a is of order $n - m < n$. This is a contradiction. ***(R1)***
Therefore $m = n$. ***(AG)***

[5 marks]

(b) Let $S(m)$ be the statement: $b^m = p^{-1} * a^m * p$.
$S(1)$ is true since we are given $b = p^{-1} * a * p$ ***(A1)***
Assume $S(k)$ as the induction hypothesis. ***(M1)***

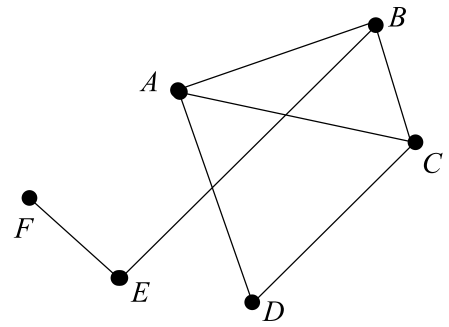
$$b^{k+1} = b^k * b = (p^{-1} * a^k * p) * (p^{-1} * a * p) \quad \textbf{\textit{(M1)}}$$
$$= p^{-1} * a^{k+1} * p \quad \textbf{\textit{(R1)}}$$

which proves $S(k+1)$.

Hence, by mathematical induction $b^n = p^{-1} * a^n * p \ (n = 1, 2, \ldots)$. ***(AG)***

[4 marks]

8. (i) (a)

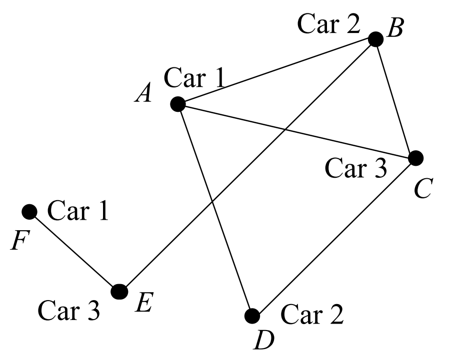


(A2)

Note:	For each error deduct one mark up to a maximum of two marks.

[2 marks]

(b) Suppose A is railroad car 1. Because B, C, and D are adjacent, they can not be in the car 1. The minimum number of railroad cars needed to transport A, B, and C is three. Putting each of these in a separate car, we draw the above graph, label the vertices, and indicate in which car the vertex will go.



Product	**Railroad car**
A	Car 1
B	Car 2
C	Car 3
D	Car 2
E	Car 3
F	Car 1

(M1)(A1)

If we use colours, then the graph can be coloured in three colours.
The chromatic number of the graph is 3. *(A1)*
Arrangement for shipping: A, F in Car 1; B, D in Car 2; and C, E in Car 3. *(A1)*

Note:	There are several possibilities. Hence there can be different graphs as well as different shipping arrangements. Please check the answers carefully and award marks as outlined above.

[4 marks]

Question 8 continued

(ii) (a) The degree sequence of this graph is given by

A	*B*	*C*	*D*	*E*	*U*
2	4	3	4	3	4

(M1)

Since degree of each vertex is not even, the graph has no Eulerian circuit. ***(R1)***

[2 marks]

(b)

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ U \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & U \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(A2)

> **Note:** For each mistake in the entries of the above matrix deduct one mark up to a maximum of two marks.

Number of walks of length 2 from vertex *A* to vertex *C* is the entry in the first row and third column of the square of the adjacency matrix.

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = 0+1+0+0+0+0=1$$

(M1)

Since this entry is 1, there is only one walk of length 2 from *A* to *C*. ***(A1)***

[4 marks]

(c) Given the graph the following steps are used to obtain the minimal spanning tree by using Kruskal's algorithm.

Let *S* be a set of vertices and *T* be a set of edges. The algorithm is organised as follows:

Choice	Edge	Weight	*T*	*S*
1	*CE*	3	*{CE}*	*{C, E}*
2	*AD*	3	*{AD, CE}*	*{C, E, A, D}*
3	*CU*	5	*{AD, CE, CU}*	*{C, E, A, D, U}*
4	*BU*	8	*{AD, CE, CU, BU}*	*{C, E, A, D, U, B}*
5	*AB*	9	*{AD, CE, CU, BU, AB}*	*{C, E, A, D, U, B}*

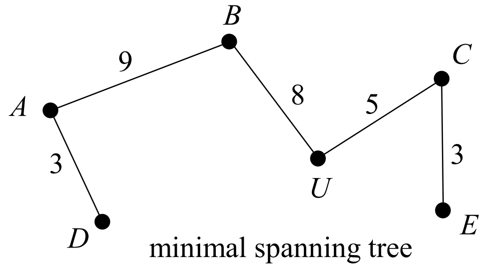
(M2)(A1)

> **Note:** If the edges and the vertices are correct so that there is a minimal spanning tree, then award ***(M2)(A1)***. For each mistake deduct one mark up to a maximum of three marks.

continued...

Question 8 (ii) (c) continued

A minimal spanning tree is



minimal spanning tree

(A2)

Total weight of minimal spanning tree is 28.

Notes:	There are other possibilities. Please read the answer carefully and award marks as outlined above. If a candidate writes only the correct answer without showing any steps, then award ***(A1)*** only.

[5 marks]

(iii) The recurrence relation can be written as $y_{n+2} - y_{n+1} - y_n = 0$.
The characteristic polynomial is $x^2 - x - 1 = 0$. Hence,

$$x = \frac{1 \mp \sqrt{5}}{2}.$$ ***(M1)***

and $$y_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$ ***(A1)***

Since $y_0 = 4$ and $y_1 = 3$, $C_1 + C_2 = 4$ and

$$C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) = 3$$ ***(M1)***

Solving for C_1 and C_2, we get

$C_1 = \frac{10+\sqrt{5}}{5}$ and $C_2 = \frac{10-\sqrt{5}}{5}$. ***(M1)(A1)***

Thus $$y_n = \left(\frac{10+\sqrt{5}}{5}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{10-\sqrt{5}}{5}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$ ***(A1)***

[6 marks]

(iv) (a) Every non-empty set of positive integers has a smallest (or least) element. ***(A2)***

(b) A set is well-ordered if any non-empty subset has a least element. ***(A2)***

(c) *Z* is not well ordered. Consider the set $S = \{x : x \le 1\}$ is a non-empty set, since 1 is in *S*. ***M1)***
If *m* is the smallest element of *S*, then $m \le 1$. Now $m - 1 \le 1$, implies $m - 1$ is in *S*. ***(A1)***
Hence *m* is not the smallest element in *S*. ***(R1)***

Note:	In answering this problem the candidates may take many different non-empty subsets like *S* above. Please examine the correctness of their argument and award marks appropriately.

[7 marks]

9. (i) (a) $f(x) = x^7 + 5x + 1$; $f'(x) = 7x^6 + 5$.

The iterates $\{x_n\}$ are given by

$$x_{n+1} = x_n - \frac{x_n^7 + 5x_n + 1}{7x_n^6 + 5} = \frac{6x_n^7 - 1}{7x_n^6 + 5},\ n = 0, 1, \ldots$$ ***(M1)***

n	x_n
0	−0.5
1	−0.20489297
2	−0.19999748
3	−0.19999744
4	−0.19999744

(A1)

> **Note:** Award ***(A1)*** for correct iterates. for any error in the calculation of iterates award ***(A0)***.

The approximate value of zero of *f* is −0.19999744 correct to eight decimal places. ***(R1)***

The third iteration x_3 is obtained from the relation

$$x_3 = \frac{6x_2^7 - 1}{7x_2^6 + 5}$$

where $x_2 = -0.19999748$. ***(A1)***

[4 marks]

(b) For fixed point iteration, $x_{n+1} = g(x_n)$, $x_0 = -0.5$, where $g(x) = x^7 + 6x + 1$. ***(M1)***

(There are other possibilities for $g(x)$)

A table of values for the iterates are given below:

n	$x_n = x_{n-1}^7 + 6x_{n-1} + 1$
0	−0.5
1	−2.0078125
2	−142.588158699
3	$-1.19835150535 \times 10^{15}$

(A1)

It does not converge. Hence this iteration will yield no zero. ***(M1)***

For $x_{n+1} = g(x_n)$ to converge, we need $|g'(x)| \le 1$.

But here $|g'(x)| = 7x^6 + 6 \ge 6 > 1$ for all values of *x*. ***(R1)***

> **Note:** There are other possibilities for the iteration. Sometimes the fixed point iteration converges.

[4 marks]

continued...

Question 9 (i) continued

(c) $f(-1) = -5 < 0$ and $f(0) = 1 > 0$. Hence, there is a zero of $f(x)$ in $(-1, 0)$ by the intermediate value theorem. ***(M1)***

Suppose there are two zeros of *f* in [– 2, 2] viz. *a* and *b*.
Hence, $f(a) = f(b) = 0$. Since *f* is a polynomial, it is differentiable and is continuous everywhere. ***(M1)***
Thus by Rolle's theorem, there is a point *c* such that $f'(c) = 0$. ***(R1)***
But $f'(c) = 7x^6 + 5 > 0$ for any *c*. Hence, *f* can not have more than one zero. ***(R1)***

[4 marks]

continued...

Question 9 continued

(ii) (a) Since, $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$,

$$\sin x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2(2n+1)}}{(2n+1)!}.$$ ***(A1)***

[1 mark]

(b) Let

$$I = \int_0^1 \sin x^2 \mathrm{d}x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \int_0^1 x^{4n+2} \mathrm{d}x$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+3)\big((2n+1)!\big)}.$$ ***(A1)***

$$= \sum_{n=0}^{\infty} (-1)^n c_n \text{, say.}$$

The sum of the alternating series $\sum_{n=0}^{\infty} (-1)^n c_n$ lies between $\sum_{n=0}^{N} (-1)^n c_n$

and $\sum_{n=0}^{N} (-1)^n c_n \pm \left| c_{N+1} \right|$.

Hence for 4 decimal place accuracy, we need $\left| c_{N+1} \right| < 0.00005$.

N	$I = \sum_{n=0}^{N} (-1)^n c_n$	$\left\| c_{N+1} \right\|$
1	$I = \frac{1}{3} - \frac{1}{7(3!)} = 0.3095238095$	$\frac{1}{11(5!)} = 0.0000757576$
2	$I = 0.3102813853$	$\frac{1}{15(7!)} = -0.000013$

(M1)(M1)

> **Notes:** Award ***(M1)*** for calculating *I* in terms of the sum.
> Award ***(M1)*** for calculating the error.

Since $-0.000013 < 0.00005$, $I = 0.3103$. ***(A1)***

[5 marks]

continued...

Question 9 continued

(iii) (a) Let $u_k(x) = \dfrac{(x-5)^k}{k^{3/2}}$.

Then $\lim\limits_{k\to\infty}\left|\dfrac{u_{k+1}(x)}{u_k(x)}\right| = \lim\limits_{k\to\infty}\dfrac{|x-5|k^{3/2}}{(k+1)^{3/2}} = |x-5|$. ***(M1)***

By the ratio test the series is convergent for $|x-5|<1$. ***(A1)***

Hence, the radius of convergence is 1. ***(R1)***

[3 marks]

(b) From part (a), the series converges absolutely for $|x-5|<1$, which is the same as $\{x : 4 < x < 6\}$. To find the interval of convergence, we have to check whether the series converges when $x = 4$ and $x = 6$. ***(M1)***

When $x = 6$, the series becomes $\sum\limits_{k=1}^{\infty} k^{-3/2}$ which converges since $\sum k^{-p}$ converges for $p > 1$. ***(R1)***

When $x = 4$, the series reduces to $\sum\limits_{k=1}^{\infty}(-1)^k k^{-3/2}$, which is an alternating series of the form $\sum\limits_{k=1}^{\infty}(-1)^k b_k$ with $b_k = k^{-3/2}$. ***(M1)***

Since, $k^{-3/2} > (k+1)^{-3/2}$, $k = 1, 2, \ldots$, and $\lim\limits_{k\to\infty} k^{-3/2} = 0$, the alternating series converges. ***(R1)***

Thus the interval of convergence is $4 \le x \le 6$. ***(R1)***

[5 marks]

(iv) Since $\sin A\cos A = \dfrac{1}{2}\sin 2A$,

$|\sin x\cos x - \sin y\cos y| = \dfrac{1}{2}|\sin 2x - \sin 2y|$. ***(M1)***

By the mean value theorem $|\sin 2x - \sin 2y| = |2x - 2y||\cos c|$ ***(M1)***

for some *c* between 2*x* and 2*y*. ***(A1)***

Since $|\cos c| \le 1$, we get $|\sin x\cos y - \sin y\cos y| \le |x - y|$. ***(R1)***

[4 marks]

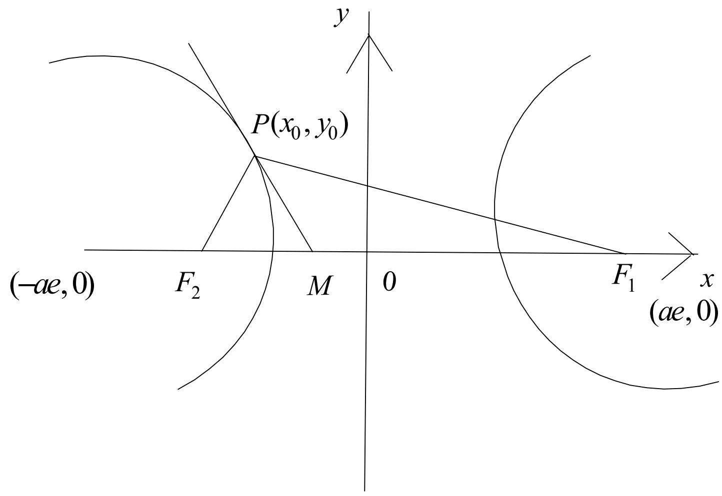
10. (i) From the given information, we have,
$b^2 = a^2\left(e_1^2 - 1\right)$, and $a^2 = b^2\left(e_2^2 - 1\right)$. ***(R1)***

Hence, $e_1^2 = \dfrac{b^2 + a^2}{a^2}$ and $e_2^2 = \dfrac{b^2 + a^2}{b^2}$. ***(M1)***

Thus, $\dfrac{1}{e_1^2} + \dfrac{1}{e_2^2} = \dfrac{b^2 + a^2}{a^2 + b^2} = 1$. ***(R1)***

[3 marks]

(ii) (a)



Equation of the hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Differentiating with respect to x, we get, $\dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0$.

Since $y' = \dfrac{b^2 x}{a^2 y}$, the slope of the tangent line (PM) at (x_0, y_0) is $\dfrac{b^2 x_0}{a^2 y_0}$. ***(M1)***

Equation of the line (PM) is

$y - y_0 = \dfrac{b^2 x_0}{a^2 y_0}(x - x_0)$ ***(M1)***

or $\dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} = \dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2} = 1$, ***(A1)***

since the point (x_0, y_0) lies on the given hyperbola.

Note:	A candidate may quote the equation directly. In that case, award ***(A1)*** for $\dfrac{xx_0}{a^2}$, ***(A1)*** for $\dfrac{yy_0}{b^2}$, and ***(A1)*** for the correct equation.

When $y = 0$, we get, $x = \dfrac{a^2}{x_0}$. Hence, the point M has coordinates $\left(\dfrac{a^2}{x_0}, 0\right)$. ***(R1)***

[4 marks]

continued...

Question 10 (ii) continued

(b) Equations of the directrices are $x=\frac{a}{e}$ and $x=-\frac{a}{e}$.
Using the fact that the distance of a point on the hyperbola from a foci is e times its distance from the directrix, we have for (x_0, y_0) on the left branch of the hyperbola, ***(M1)***

$$\text{PF}_1=\left|e\left(-x_0+\frac{a}{e}\right)\right|=|-ex_0+a|=|ex_0-a|$$ ***(A1)***

and

$$\text{PF}_2=\left|e\left(-x_0-\frac{a}{e}\right)\right|=|-ex_0-a|=|ex_0+a|.$$ ***(A1)***

Also, $\text{MF}_1=\left|ae-\frac{a^2}{x_0}\right|$ ***(A1)***

and $\text{MF}_2=\left|ae+\frac{a^2}{x_0}\right|$ ***(A1)***

The same results are obtained if (x_0, y_0) is on the right branch of the hyperbola.

OR

$$\text{PF}_1^2=(x_0-ae)^2+y_0^2=x_0^2-2aex_0+a^2e^2+y_0^2$$

Since (x_0, y_0) lies on the hyperbola,

$$y_0^2=\frac{b^2x_0^2}{a^2}-b^2=x_0^2(e^2-1)-a^2(e^2-1)$$

Therefore, $\text{PF}_1^2=x_0^2-2aex_0+a^2e^2+x_0^2(e^2-1)-a^2(e^2-1)$ ***(M1)***
$=x_0^2-2aex_0+a^2$
$=(x_0e-a)^2$

Hence, $\text{PF}_1=|x_0e-a|$. ***(A1)***

Similarly, $\text{PF}_2=|x_0e+a|$. ***(A1)***

Hence, $\text{MF}_1=\left|ae-\frac{a^2}{x_0}\right|$ ***(A1)***

and

$$\text{MF}_2=\left|ae+\frac{a^2}{x_0}\right|.$$ ***(A1)***

[5 marks]

continued...

Question 10 (ii) continued

(c) We shall show that $\frac{\text{PF}_1}{\text{PF}_2} = \frac{\text{MF}_1}{\text{MF}_2}$ and use the converse of the bisector theorem to conclude that [PM] is the angle bisector of $\text{F}_1\hat{\text{P}}\text{F}_2$. ***(M1)***

EITHER $\text{PF}_1 \times \text{MF}_2 = \left|(-ex_0 + a)\left(ae + \frac{a^2}{x_0}\right)\right|$

$$= \left|\left(-ae^2x_0 + a^2e - a^2e + \frac{1}{2}\frac{a^3}{x_0}\right)\right| \qquad \textit{(M1)}$$

and

$$\text{PF}_2 \times \text{MF}_1 = \left|(-ex_0 - a)\right|\left|\left(ae - \frac{a^2}{x_0}\right)\right|$$

$$= \left|\left(-ae^2x_0 - a^2e + a^2e + \frac{1}{2}\frac{a^3}{x_0}\right)\right| \qquad \textit{(M1)}$$

Hence $\text{PF}_1 \times \text{MF}_2 = \text{PF}_2 \times \text{MF}_1$. ***(R1)***

OR $\frac{\text{PF}_1}{\text{PF}_2} = \frac{|x_0e - a|}{|x_0e + a|}$ ***(A1)***

and

$$\frac{\text{MF}_1}{\text{MF}_2} = \left|\frac{ae - (a^2/x_0)}{ae + (a^2/x_0)}\right| \qquad \textit{(M1)}$$

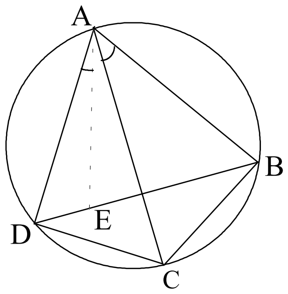
$$= \left|\frac{x_0e - a}{x_0e + a}\right| = \frac{\text{PF}_1}{\text{PF}_2} \qquad \textit{(A1)}$$

[4 marks]

continued...

Question 10 continued

(iii) Ptolemy's theorem: If ABCD is a cyclic quadrilateral, then $AB \times DC + BC \times AD = AC \times BD$.



Draw [AE] with E on [DB] so that $D\hat{A}E = C\hat{A}B$,
ΔADE and ΔABC are similar because $D\hat{A}E = B\hat{A}C$,
and $A\hat{D}E = A\hat{C}B$ (being on the same arc of a circle). ***(M1)***

Hence, $\frac{AD}{DE} = \frac{AC}{CB}$ or $AD \times CB = AC \times DE$. (1) ***(R1)***

Similarly, ΔADC and ΔABE are similar, since
$D\hat{C}A = E\hat{B}A$ and $D\hat{A}C = D\hat{A}E + E\hat{A}C = B\hat{A}C + E\hat{A}C = B\hat{A}E$ ***(M1)***

Hence, $\frac{DC}{CA} = \frac{EB}{AB}$ or $DC \times AB = EB \times AC$. (2) ***(R1)***

Add (1) and (2) to get

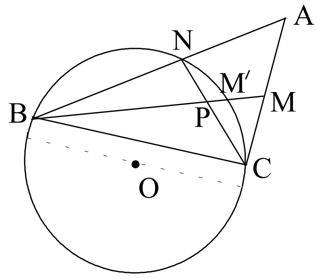
$AD \times CB + DC \times AB = AC(DE + EB) = AC \times DB$. ***(R1)***

[5 marks]

continued...

Question 10 continued

(iv) (a)



To show that the quadrilateral BNM′C is cyclic, it is enough to show that $B\hat{N}C = B\hat{M}'C$

In ΔNBP, $N\hat{B}P + B\hat{N}C + N\hat{P}B = 180°$ (1) ***(M1)***

In $\Delta MC'P$, $M'\hat{C}P + C\hat{M}'P + M'\hat{P}C = 180°$ (2) ***(M1)***

Equating (1) and (2), and using $N\hat{B}P = M'\hat{C}N = M'\hat{C}P$ and $N\hat{P}B = M'\hat{P}C$, ***(M1)***

we have, $B\hat{N}C = C\hat{M}'P = C\hat{M}'B$. ***(A1)***

Hence BNM′C is a cyclic quadrilateral. ***(AG)***

OR

M′ is the point on [BM] nearer M such that

$M'\hat{C}N = \frac{1}{2}A\hat{B}C$. ***(C1)***

Since [BM] is the angle bisector of $A\hat{B}C$, $M'\hat{B}N = \frac{1}{2}A\hat{B}C$. ***(M1)***

Hence, $M'\hat{B}N = M'\hat{C}N$. ***(R1)***

Since these are angles on arc M′N, BNM′C is a cyclic quadrilateral. ***(R1)(AG)***

[4 marks]

(b) By part (a) there is a circle on which the points B, N, M′, and C lie. [BN] and [BM′] are chords of this circle.

Since, $A\hat{C}B > A\hat{B}C = N\hat{B}C, \frac{1}{2}A\hat{C}B > \frac{1}{2}N\hat{B}C$ ***(M1)***

or $N\hat{C}B > M'\hat{B}C$. ***(A1)***

Hence, $N\hat{C}B + \frac{1}{2}A\hat{B}C > M'\hat{B}C + \frac{1}{2}A\hat{B}C$ ***(M1)***

or $M'\hat{C}B > N\hat{B}C$, since $N\hat{B}C = A\hat{B}C$. ***(A1)***

In the circle BNM′C, we use the result that if two inscribed angles intercept unequal chords, the greater angle intercepts the greater chord, we get $BM' > CN$. ***(R1)***

[5 marks]